

NASA/TM-2000-210123  
ARL-TR-2206



# Determining a Prony Series for a Viscoelastic Material From Time Varying Strain Data

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May 2000

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May 2000

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# Determining a Prony Series for a Viscoelastic Material from Time Varying Strain Data

## ABSTRACT

In this study a method of determining the coefficients in a Prony series representation of a viscoelastic modulus from rate dependent data is presented. Load versus time test data for a sequence of different rate loading segments is least-squares fitted to a Prony series hereditary integral model of the material tested. A nonlinear least squares regression algorithm is employed. The measured data includes ramp loading, relaxation, and unloading stress-strain data. The resulting Prony series, which captures strain rate loading and unloading effects, produces an excellent fit to the complex loading sequence.

**KEY WORDS:** hereditary integral, viscoelasticity, weighted nonlinear regression, Prony series, multiple loading segments

## INTRODUCTION

In order to determine the time dependent stress - strain state in a linear viscoelastic material, under an arbitrary loading process, the deformation history must be considered. The time dependent constitutive equations of a solid viscoelastic material include these history effects. The load (stress) and displacement (strain) history, the loading rate (displacement rate), and time of load application on the specimen are all needed to determine the constants in the constitutive equations. A common form for these constitutive equations

employs a Prony series (i.e., a series of the form  $\sum_{i=1}^N \alpha_i \cdot e^{-t/\tau_i}$ ).

Creep and relaxation tests are most commonly used to determine the viscoelastic material properties, see Figure 1. In ideal relaxation and creep tests, the displacements or loads are applied to the specimen instantly. In the real test, especially for a large structural component, limitations of the testing equipment result in a relatively low strain rate and long

period of loading. The response during the period of loading is typically ignored, and only the data obtained during the period of constant displacement or constant load are used to determine the material properties.<sup>1,2</sup> Ignoring this long loading period and the strain rate effects in the data reduction introduces additional errors in the determination of the material properties.

There are numerous methods for determining the Prony series from relaxation and/or creep data. An early method<sup>3</sup> involved constructing log-log plots of relaxation data in which straight line approximations for the data on the log-log graph yield the time constants (i.e.  $\tau_i$ 's) from the slopes, and the exponential coefficients (i.e.  $\alpha_i$ 's) are obtained from the intercepts. Other methods have also been employed. For example, Johnson and Quigley<sup>4</sup> determined a relaxation time constant for a nonlinear model which is similar to a one-term Prony series model. They minimized a least-squares error measure, of the difference between the nonlinear model and measured data, by iteratively integrating (numerically) an internal variable equation. When attempting to determine relaxation time constants for higher fidelity nonlinear models, Johnson, et al.<sup>5</sup> employed trial and error procedures, similar to early linear methods,<sup>3</sup> due to the complexity of the resulting nonlinear least-squares problem. More recently, a few authors<sup>6-8</sup> employed nonlinear optimization methods to obtain a high quality Prony series representation of relaxation data with a minimum number of terms in the series. The viscoelastic model can also be formulated in differential form. This is becoming popular recently<sup>9-11</sup> since the differential models can be effectively incorporated into finite element algorithms. When using these internal variable methods, each Prony series term is associated with a material internal state variable. In the discrete finite element model, each term in the Prony series adds a substantial number of global variables. Thus, a short Prony series, which can accurately represent the material, is desirable. Nonlinear regression methods can help with determining a short and accurate Prony series.

The purpose of this paper is to present a method for including the loading and unloading data, along with the relaxation data, in a nonlinear regression analysis to obtain the Prony series. The resulting viscoelastic material model is then capable of simulating the loading segments as well as the relaxation segments. This is an improvement when modeling hysteretic effects is important. The analytical solution for loading and/or unloading is determined herein and employed in a nonlinear regression analysis to determine the Prony series. In addition, data weighting functions are investigated and are shown to improve the

fit in the beginning of relaxation period. Again, this method allows all the measured data to be included and results in an improved constitutive model.

## Hereditary Integrals for linear viscoelasticity

Detailed descriptions of linear viscoelasticity can be found in the literature.<sup>1,2</sup> An overview of linear viscoelasticity is provided here in order to introduce the hereditary integral method which is used below to determine the analytical solution for the loading and unloading segments. Linear viscoelastic constitutive models are represented by simple physical models composed of springs and dashpots. The spring is the linear-elastic component, and its constitutive equation is

$$\sigma = E \cdot \varepsilon \quad (1)$$

The dashpot is the viscous component, and its constitutive equation is

$$\sigma = \eta \cdot \frac{\partial \varepsilon}{\partial t} \quad (2)$$

where  $\eta$  is the viscosity constant. Linear viscoelastic constitutive models are constructed by superimposing components with constitutive equations given by equations (1) and (2). Since the mechanical response of the dashpot is time dependent, the behavior of a viscoelastic material that is modeled by parallel and/or series combinations of springs and dashpots is also time dependent.

The creep test consists of a constant stress,  $\sigma_0$ , applied to a specimen for a period of time while its strain is recorded (Figure 1a). In a relaxation test, the specimen's strain,  $\varepsilon_0$ , is held constant for a period of time while the stress is recorded (Figure 1b). In Figure 1,  $\varepsilon_0$  and  $\sigma_0$  are the initial strain and stress, respectively. For the relaxation test, a constitutive relation for the period of constant strain can be written as follows:

$$\sigma(t) = Y(t) \cdot \varepsilon_0 \quad (3)$$

where  $Y(t)$  is a relaxation function. When the material is assumed to be a general Maxwell solid, the relaxation function is typically modeled with a Prony series as follows,

$$Y(t) = E_0 \cdot \left(1 - \sum_{i=1}^n p_i \cdot (1 - e^{-t/\tau_i})\right) \quad (4)$$

where:

$p_i$  is the  $i$ 'th Prony constant ( $i = 1, 2, \dots$ )

$\tau_i$  is the  $i$ 'th Prony retardation time constant ( $i = 1, 2, \dots$ )

$E_0$  is the instantaneous modulus of the material

For time  $t = 0$ ,  $Y(0) = E_0$  and for  $t = \infty$ ,  $Y(\infty) = E_0 (1 - \sum p_i)$ .

In the case of a creep test, a creep compliance function,  $J(t)$ , is defined as follows.

$$\varepsilon(t) = J(t) \cdot \sigma_0 \quad (5)$$

The compliance function is then determined by procedures analogous to those described above.

To determine the stress state in a viscoelastic material at a given time, the deformation history must be considered. For linear viscoelastic materials, a superposition of hereditary integrals describes the time dependent response<sup>1</sup>. If a specimen is load free prior to the time  $t = 0$ , at which a stress,  $\sigma_0 + \sigma(t)$ , is applied the strain for time  $t > 0$  can be represented as follows.

$$\varepsilon(t) = \sigma_0 \cdot J(t) + \int_0^t J(t - \xi) \frac{d\sigma(\xi)}{d\xi} d\xi \quad (6)$$

where  $J(t)$  is the compliance function of the material and  $d\sigma(\xi)/d\xi$  is the stress rate. A similar equation can be used for the relaxation model to obtain the stress function introduced by an arbitrary strain function<sup>1</sup>

$$\sigma(t) = \varepsilon_0 \cdot Y(t) + \int_0^t Y(t - \xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \quad (7)$$

where  $Y(t)$  is the relaxation function (Equation 4) and  $d\varepsilon(\xi)/d\xi$  is the strain rate. An example of applying hereditary integrals for a multiple loading segment process is shown in next section.

## Hereditary integrals for a multiple loading process

Hereditary integrals with Prony series kernels can be applied to model a loading process such as the one shown in Figure 2. The process in Figure 2 is divided into four segments for which strain and strain rate functions are defined. The functions are:

$$\varepsilon(t) = \begin{cases} \varepsilon_1 \cdot t / (t_1 - t_0) & t_0 < t \leq t_1 \\ \varepsilon_1 & t_1 < t \leq t_2 \\ -\varepsilon_1 \cdot t / (t_3 - t_2) & t_2 < t \leq t_3 \\ 0 & t_3 < t \leq t_4 \end{cases}, \quad \frac{d\varepsilon}{dt} = \begin{cases} \varepsilon_1 / (t_1 - t_0) & t_0 < t \leq t_1 \\ 0 & t_1 < t \leq t_2 \\ -\varepsilon_1 / (t_3 - t_2) & t_2 < t \leq t_3 \\ 0 & t_3 < t \leq t_4 \end{cases} \quad (8)$$

where  $\varepsilon_0 = \varepsilon(0) = 0$  and  $t_0 = 0$ .

For a material with a relaxation function in the form of a Prony series (Equation 4), the stress functions of the loading process can be derived as follows:

### Step 1. ( $t_0 < t \leq t_1$ )

Substitute Equations 4 and first strain and strain rate functions of Equation 8 into Equation 7 and obtain:

$$\begin{aligned} \sigma(t) &= \varepsilon_0 \cdot Y(t) + \int_0^t Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \\ &= 0 + \int_0^t E_0 \cdot \left(1 - \sum_{i=1}^n p_i \cdot (1 - e^{-\xi/\tau_i})\right) \cdot \frac{\varepsilon_1}{t_1} \cdot d\xi \\ &= \frac{E_0 \cdot \varepsilon_1}{t_1} \cdot \left[ \xi - \sum p_i \xi + \sum p_i \cdot \tau_i \cdot e^{-\xi/\tau_i} \right]_0^t \\ &= \frac{E_0 \cdot \varepsilon_1}{t_1} \cdot \left[ t - \sum p_i t + \sum p_i \cdot \tau_i - \sum p_i \cdot \tau_i \cdot e^{-t/\tau_i} \right] \end{aligned} \quad (9)$$

where n is the number of terms in the Prony series. To simplify the expression, n will not be shown in following equations.  $p_i$  and  $\tau_i$  are the constants in the i-th term of the Prony series.

### Step 2. ( $t_1 < t \leq t_2$ )

Using the second strain rate function obtain:

$$\begin{aligned}
\sigma(t) &= \varepsilon_0 \cdot Y(t) + \int_0^{t_1^-} Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi + \int_{t_1^+}^t Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \\
&= 0 + \frac{E_0 \cdot \varepsilon_1}{t_1} \left[ \xi - \sum p_i \xi + \sum p_i \tau_i \cdot e^{-\frac{(t-\xi)}{\tau_i}} \right]_0^{t_1} + 0 \\
&= \frac{E_0 \cdot \varepsilon_1}{t_1} (t_1 - \sum p_i t_1 + \sum p_i \tau_i \cdot e^{-\frac{(t-t_1)}{\tau_i}} - \sum p_i \tau_i \cdot e^{-\frac{t}{\tau_i}})
\end{aligned} \tag{10}$$

**Step 3.** ( $t_2 < t \leq t_3$ )

The third strain rate function yields:

$$\begin{aligned}
\sigma(t) &= \varepsilon_0 \cdot Y(t) + \int_0^{t_1^-} Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi + \int_{t_1^+}^{t_2^-} Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \\
&\quad + \int_{t_2^+}^t Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \\
&= 0 + \frac{E_0 \cdot \varepsilon_1}{t_1} \left[ \xi - \sum p_i \xi + \sum p_i \tau_i \cdot e^{-\frac{(t-\xi)}{\tau_i}} \right]_0^{t_1} + 0 \\
&\quad - \frac{E_0 \cdot \varepsilon_1}{(t_3 - t_2)} \left[ \xi - \sum p_i \xi + \sum p_i \tau_i \cdot e^{-\frac{(t-\xi)}{\tau_i}} \right]_{t_2}^t \\
&= \frac{E_0 \cdot \varepsilon_1}{t_1} (t_1 - \sum p_i t_1 + \sum p_i \tau_i \cdot e^{-\frac{(t-t_1)}{\tau_i}} - \sum p_i \tau_i \cdot e^{-\frac{t}{\tau_i}}) \\
&\quad - \frac{E_0 \cdot \varepsilon_1}{(t_3 - t_2)} (t - \sum p_i t + \sum p_i \tau_i - t_2 + \sum p_i t_2 - \sum p_i \tau_i \cdot e^{-\frac{(t-t_2)}{\tau_i}})
\end{aligned} \tag{11}$$

Note, the first portion of Equation 11 is equal to Equation 10. Thus,

$$\sigma(t) = \sigma_2(t) - \frac{E_0 \cdot \varepsilon_1}{(t_3 - t_2)} (t - \sum p_i t + \sum p_i \tau_i - t_2 + \sum p_i t_2 - \sum p_i \tau_i \cdot e^{-\frac{(t-t_2)}{\tau_i}}) \tag{12}$$

where  $\sigma_2(t)$  is Equation 10 and it is function of time.

#### Step 4. ( $t_3 < t \leq t_4$ )

Similarly, the equation for the fourth step can be written as follows:

$$\begin{aligned} \sigma(t) = & \sigma_2(t) - \frac{E_0 \cdot \varepsilon_1}{(t_3 - t_2)} (t_3 - \sum p_i t_3 + \sum p_i \tau_i \cdot e^{-\frac{-(t-t_3)}{\tau_i}} - t_2 \\ & + \sum p_i t_2 - \sum p_i \tau_i \cdot e^{-\frac{-(t-t_2)}{\tau_i}}) \end{aligned} \quad (13)$$

A numerical example of a multiple loading segment process using MATHCAD<sup>12</sup> software is shown in Appendix A. In the example, the stress function was calculated based on the strain and strain rate functions shown above, and it employed a two-term Prony series. The results of the viscoelastic analysis are shown in the stress-time and stress-strain plots. This worksheet can be used to generate data in a parametric study involving viscoelastic materials. The worksheet is also used as part of the weighted nonlinear regression algorithm as is shown in the following sections.

### Weighted Nonlinear Regression

The Prony series coefficients and retardation times appearing in Equation 4 need to be determined in a regression analysis. Here, a standard nonlinear regression method (the Marquardt-Levenberg Method<sup>13,14</sup>) is used to perform the data fitting. In the nonlinear regression, an error function ( $\chi^2$ ) with respect to the unknown constants is defined as,

$$\chi^2(\mathbf{a}) = \sum_{i=1}^N \left[ \frac{y_i - y(x_i; \mathbf{a})}{\sigma_i} \right]^2 \quad (14)$$

where  $x_i$  and  $y_i$  are the experimental data, function  $y(x_i; \mathbf{a})$  is the model to be fitted, and  $\sigma_i$  is the standard deviation of measurement error of  $i$ -th data point. A set of unknown constants ( $\mathbf{a}$ ) will be determined that minimize the error function  $\chi^2$ . The error function ( $\chi^2$ ) is approximated by its Taylor series with the quadratic form:

$$\chi^2(\mathbf{a}) \approx c - \mathbf{d} \cdot \mathbf{a} + \frac{1}{2} \cdot \mathbf{a} \cdot \mathbf{D} \cdot \mathbf{a} \quad (15)$$

where  $c$  is a constant and  $\mathbf{d}$  is the gradient of  $\chi^2$  with respect to the parameters  $\mathbf{a}$ , which will be zero where  $\chi^2$  is minimum. Matrix  $\mathbf{D}$  is the second partial derivative of  $\chi^2$  with respect to the parameters  $\mathbf{a}$ . Initial trial values of  $\mathbf{a}$  are specified and improved values are determined

by the nonlinear regression algorithm. Iteration is continued until the error function,  $\chi^2$ , effectively stops decreasing.

Since each Prony term includes two variables ( $p_i$  and  $\tau_i$ ) and since the instantaneous modulus ( $E_0$ ) must be determined, the total number of variables in the regression is  $2n+1$ . Based on thermodynamic principles, several constraint conditions must be applied:

$$P_i \geq 0, \quad \sum P_i \leq 1, \quad \tau_i \geq 0, \quad E_0 > 0 \quad (16)$$

In addition, the distribution of the standard deviation of measurement error ( $\sigma$ ) is not easily determined based on the error of data acquisition equipment and the error of test machine, the error is usually assumed to be uniform for all data points ( $\sigma_i = 1$ ). As is well known, the viscoelastic effects are most significant at the beginning of the relaxation period, the fitting error in this region is significant. Since the percentage of the number of data points at the beginning of the relaxation period is less, the error function  $\chi^2$  is dominated by a long uniform tail region of the relaxation period. To reduce the error and improve the fit at beginning of the relaxation period, a weight function ( $w = 1/\sigma$ ,  $0 < \sigma < 1$ ) is used. The larger the weight factor at a data point, the better the curve fit the data point. There is no analytical method to determine the weight function, thus a trial-and-error method is used. The acceptance of the weight function is based on a graph of the data and regression model results. Weighted nonlinear regression requires more iterations than unweighted nonlinear regression, but it can provide a better fit to the experimental data in the region of most interest.

## **Weighted Nonlinear Regression for Relaxation Test**

A three-point bending relaxation test of a composite material was performed<sup>15</sup>. The specimen (12 in. x 2 in. x 0.768 in.) was loaded in 22 seconds to a maximum deflection of 0.103 in. at the middle of the span. Then the deflection was held for 11,711 seconds (Figure 3). The Marquardt-Levenberg nonlinear regression method was applied by the commercial software SigmaPlot<sup>13</sup> to a hereditary integral model using two segments (loading and holding) to obtain the Prony series coefficients for the data. Since no analytical method exists to form the weight function, a trial-and-error method based on material properties was used to obtain a fit curve. The viscoelastic stress decays exponentially in the relaxation test,

therefore the viscoelastic effect is more significant during the loading period and at the beginning (< 30 seconds) of the holding period. The number of data points in these periods (<100) is much less than the number of data points in tail region of the relaxation period (> 5000). The error function ( $\chi^2$ ) will be dominated by a long uniform tail region of the relaxation period if a uniform weight function is applied. Therefore, a piecewise weight function was used to obtain better fits for these periods and improve the accuracy of the regression. Figure 4 shows the load relaxation at the beginning of the process. The dots represent the test data. Three regression results are shown. The dash-dot curve is the result of a regression analysis without the weight function (w/o WF) for a two-term Prony series. The long-dash curve is the result for a two-term Prony series with weight function number 1 (WF1) shown below:

$$w = \begin{cases} 150 \leq t < 22 \\ 10022 \leq t \leq 50 \\ 150 < t \leq 11711 \end{cases} \quad (17)$$

The short-dash curve is the result for a three-term Prony series with weight function 2 (WF2) as follows:

$$w = \begin{cases} 10 \cdot t0 \leq t < 22 \\ 10^6 / (t^2) 22 \leq t \leq 1000 \\ 11000 < t \leq 11711 \end{cases} \quad (18)$$

Since an initial value is required for each of the variables, a trial data set based on the test data was assumed. The sum of the P constants should be about 0.09 since the load at the end of relaxation period is 9% lower than the load at the beginning. The retardation time constants can be set to powers of ten. As long as the initial trial values are reasonable, convergence will be achieved.

As is well known, the regression data fits better in a particular region if the relative weight factor (weight factor / sum of weight factors) in that region is greater than the average value. In Figure 4, the curves with weighted functions were closer to the data points near the beginning of the relaxation period. Since the weight factor at  $t = 22$  seconds of function 2 (=2066) is greater than the value of the weight function 1 (=100), the curve of function 2 is a better match to the data than curve of function 1 at the beginning of relaxation.

However, as shown in the Figure 3, weight function 2 does not fit the data as well as the other curves after 1000 seconds. It appears that function 2 was over weighted at the beginning of the relaxation and the relative weight factor at the other region was too small.

The Prony constants of the regression are shown in Table 1.

**Table 1.** The Prony constants for the regression

	Modulus ( $E_0$ )	$P_1$	$\tau_1$ (sec. )	$P_2$	$\tau_2$ (sec. )	$P_3$	$\tau_3$ (sec. )
w/o WF	7259.4	0.0259	134.69	0.0507	4898.64		
WF 1	7323.8	0.0319	57.23	0.0502	4040.36		
WF 2	7450.9	0.0262	9.29	0.0180	80.65	0.0460	2028.03

The results showed that a weight function should be selected based on the material properties and test data distribution. Properly selecting the weight function in the most significant region can improve accuracy of the regression.

### **Weighted Nonlinear Regression for a Multiple Loading Process**

In a recent study<sup>15</sup>, a thick composite panel responded viscoelastically when it was tested. In order to characterize the properties of the panel, a three-point bending test with multiple loading segment processes was performed on it. Since the stiffness of the panel was quite high, the test was conducted with a large hydraulic testing machine. High deformation rates were not available with this loading machine. The loading head displacement schedule of the machine is shown in Figure 5. The schedule is unlike a standard relaxation test. Though load relaxation periods exist, the time required to apply a full load was clearly long when compared to the relaxation periods. As mentioned in the previous sections, in order to include the data of loading and unloading segments, a nonlinear regression combined with the hereditary integral was used to simulate the data.

Since the deformation was very small compared to the specimen dimensions a linear model was used for the calculations. The resulting displacements and the loads are linearly related to the strains and stresses. Thus, the displacement and load data were used as strain and stress data in the equations.

The displacement (strain) schedule in Figure 5 was divided into 11 steps. The piecewise continuous function for the first 10 steps was assumed to be approximately linear, similar to the previous example (Equation 8). The data for step 11 was fit to a cubic

polynomial for a more precise regression model. The strain and strain rate functions were defined as follows,

$$\varepsilon(t) = \begin{cases} \varepsilon_1 \cdot t / (t_1 - t_0) & t_0 < t \leq t_1 \\ \varepsilon_1 & t_1 < t \leq t_2 \\ \dots & \dots \\ \varepsilon_i + a \cdot t + b \cdot t^2 + \dots & t_{10} < t \leq t_{11} \end{cases} \quad (19)$$

$$\frac{d\varepsilon}{dt} = \begin{cases} \varepsilon_1 / (t_1 - t_0) & t_0 < t \leq t_1 \\ 0 & t_1 < t \leq t_2 \\ \dots & \dots \\ a + 2 \cdot b \cdot t + \dots & t_{10} < t \leq t_{11} \end{cases}$$

The parameters of the polynomial used in step 11 were  $\varepsilon_i = -6.47997$ ,  $a = 0.022976$ ,  $b = -2.516844e-5$  and  $c = 8.772892e-9$ .

The piecewise continuous stress function for first 10 steps, based on the hereditary integral, was derived by the same technique as shown from Equation 9 to Equation 13. The load function for step 11 was derived as follows:

$$\begin{aligned} \sigma(t) &= \sigma_{10}(t) + \int_{t_{10}}^t Y(t-\xi) \frac{d\varepsilon(\xi)}{d\xi} d\xi \\ &= \sigma_{10}(t) + \int_{t_{10}}^t E_0 \left(1 - \sum_{i=1}^n p_i (1 - e^{-(t-\xi)/\tau_i})\right) \cdot (a + 2 \cdot b \cdot \xi + 3 \cdot c \cdot \xi^2) \cdot d\xi \\ &= \sigma_{10}(t) + E_0 \cdot [(a \cdot \xi + b \cdot \xi^2 + c \cdot \xi^3) - \sum p_i \cdot (a \cdot \xi + b \cdot \xi^2 + c \cdot \xi^3) \\ &\quad + \sum p_i \cdot e^{-(t-\xi)/\tau_i} \cdot \tau_i \cdot (a + 2 \cdot b \cdot (\xi - \tau_i) + 3 \cdot c \cdot (\xi^2 - 2 \cdot \xi \cdot \tau_i + 2 \cdot \tau_i^2))]_{t_{10}}^t \\ &= \sigma_{10}(t) + E_0 \cdot [(1 - \sum p_i) \cdot (a \cdot (t - t_{10}) + b \cdot (t^2 - t_{10}^2) + c \cdot (t^3 - t_{10}^3)) \\ &\quad + \sum p_i \cdot \tau_i \cdot (a + 2 \cdot b \cdot (t - \tau_i) + 3 \cdot c \cdot (t^2 - 2 \cdot t \cdot \tau_i + 2 \cdot \tau_i^2)) \\ &\quad - \sum p_i \cdot e^{-(t-t_{10})/\tau_i} \cdot \tau_i \cdot (a + 2 \cdot b \cdot (t_{10} - \tau_i) + 3 \cdot c \cdot (t_{10}^2 - 2 \cdot t_{10} \cdot \tau_i + 2 \cdot \tau_i^2))] \end{aligned} \quad (20)$$

where  $\sigma_{10}(t)$  is the stress function from step 10.

A piecewise weight function was generated in order to increase the accuracy of regression. Again, the weight function was determined by a trial-and-error method. After

several iterations, the constant weight factors used for each step were determined as:  $W(t) = [1.0, 5.0, 1.0, 1.5, 1.0, 7.0, 1.0, 1.5, 1.0, 5.0, 1.0]$  and the analysis results were shown in Figure 6. The weight factors were equal to 1.0 for the loading and unloading steps, and were greater than 1.0 for the holding steps.

The total number of variables in the Prony series material model is  $2n+1$ , where  $n$  is the number of Prony terms. In this study, a two-term Prony series was sufficient to fit the data. The regression analysis results and test data are shown in Figures 6 and 7. The only difference between the results of the two analyses, shown in the figures, was the displacement function of step 11. One of the analyses assumed that the displacement function is linear and other assumed it was cubic. Both model results agreed closely with the experimental data. Since the cubic polynomial described displacement in step 11 more precisely, the one with cubic displacement function fit better in step 11 than the one with linear function. The error function ( $\chi^2$ ) of the one with cubic function is 10% lower than the one with linear function.

## **CONCLUDING REMARKS:**

A method of determining the coefficients in a Prony series representation of a viscoelastic modulus from rate dependent data has been presented. The hereditary integral method was employed to obtain an analytical representation of material response when it is subjected to rate dependent loading. The analytical representation was used in a nonlinear regression analysis, with measured data, to evaluate the Prony series constants. Several regression analyses were performed using different weight functions. For the data analyzed in this study, improved simulations of the hysteresis effects were obtained when the data at the beginning of each relaxation period was appropriately weighted. Note, the data analyzed here had loading and relaxation regions of similar length in time. Other weighting functions may be needed for different loading schedules.

The method presented here provided a highly accurate representation of the material behavior in the rate dependent loading region. It can also represent the response of a viscoelastic material for other unique loading schedules. For example, it can be used for schedules in which the material is not allowed to relax between subsequent loading changes.

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## Appendix A Numerical Solution for Multiple Loading Segment Process

The purpose of this appendix is to present an input file for MATHCAD which can be used to numerically compute results from Equations (9) to (13) for the multiple loading segment process. Two Prony terms are used in this simulation. The loading process is defined by Figure 2. In the data file below, text with a bold font represents a comment and with a normal font represents a command. The result of the simulation is saved to a file (OUTPUT.PRN) which is shown at end of the appendix.

### Input file (Hereditary.MCD)

The relaxation function of the material is as follows:

$$Y(t) = E ( 1 - P1(1 - e^{-t/\tau1}) - P2 (1-e^{-t/\tau2}))$$

Viscoelastic Material Constants:

$$E := 10^9 \quad P1 := 0.2 \quad \tau1 := 10 \quad P2 := 0.1 \quad \tau2 := 100$$

where E is the modulus, P and  $\tau$  are Prony constants

Loading time: (Seconds)

$$\begin{aligned} \text{delt} &:= 5 & \text{holdt} &:= 50 & t0 &:= 0 & t1 &:= t0 + \text{delt} & t2 &:= t1 + \text{holdt} \\ & & t3 &:= t2 + \text{delt} & & & t4 &:= t3 + \text{holdt} \end{aligned}$$

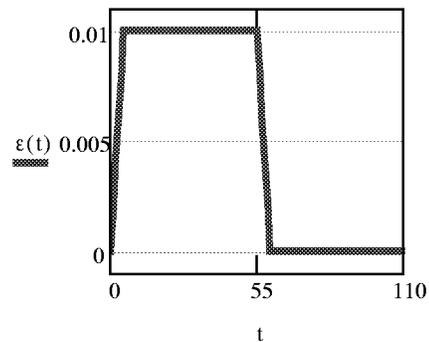
Piecewise Strain Function: (Multiple Loading Process)

$$\epsilon1 := 0.01 \quad \epsilon2 := 0.0$$

$$t := t0 . . t4$$

strain function

$$\epsilon(t) := \begin{cases} \left[ \epsilon1 \cdot \frac{t}{(t1 - t0)} \right] & \text{if } 0 < t \leq t1 \\ (\epsilon1) & \text{if } t1 < t \leq t2 \\ \left[ \epsilon1 + (\epsilon2 - \epsilon1) \cdot \frac{t - t2}{(t3 - t2)} \right] & \text{if } t2 < t \leq t3 \\ (\epsilon2) & \text{if } t3 < t \leq t4 \\ 0 & \text{otherwise} \end{cases}$$



Hereditary Integral for the stress function

Step 1 (  $t0 < t < t1$  ) Loading to  $\epsilon = \epsilon1$

$$t := t0..t1$$

$$\sigma1(t) := E \cdot \frac{\epsilon1}{t1} \cdot \left( t - P1 \cdot t + P1 \cdot \tau1 - P1 \cdot \tau1 \cdot \exp\left(-\frac{t}{\tau1}\right) - P2 \cdot t + P2 \cdot \tau2 - P2 \cdot \tau2 \cdot \exp\left(-\frac{t}{\tau2}\right) \right)$$

Step 2 (  $t1 < t < t2$  ) Holding the load for 50 seconds

$$t := t1..t2$$

$$\begin{aligned} \sigma2(t) := E \cdot \frac{\epsilon1}{t1} \cdot & \left( t1 - P1 \cdot t1 + P1 \cdot \tau1 \cdot \exp\left(-\frac{t-t1}{\tau1}\right) - P1 \cdot \tau1 \cdot \exp\left(-\frac{t}{\tau1}\right) - P2 \cdot t1 \right. \\ & \left. + P2 \cdot \tau2 \cdot \exp\left(-\frac{t-t1}{\tau2}\right) - P2 \cdot \tau2 \cdot \exp\left(-\frac{t}{\tau2}\right) \right) \end{aligned}$$

**Step 3 (t2<t<t3) Unload to ε=ε2**

t := t2..t3

$$\sigma_2(t) := \frac{E \cdot (\epsilon_2 - \epsilon_1)}{t_3 - t_2} \cdot (t - P_1 \cdot t + P_1 \cdot \tau_1 - P_2 \cdot t + P_2 \cdot \tau_2 - t_2 + P_1 \cdot t_2 + P_2 \cdot t_2 - P_1 \cdot \tau_1 \cdot \exp(-\frac{t-t_2}{\tau_1}) - P_2 \cdot \tau_2 \cdot \exp(-\frac{t-t_2}{\tau_2}))$$

$\sigma_3(t) := \sigma_2(t) + \sigma_{3\_1}(t)$

**Step 4 ( t3 < t < t4) Holding for 50s**

t := t3, t3 + 5..t4

$$\sigma_{4\_1}(t) := \frac{E \cdot (\epsilon_2 - \epsilon_1)}{(t_3 - t_2)} \cdot (t_3 - P_1 \cdot t_3 + P_1 \cdot \tau_1 \cdot \exp(-\frac{t-t_3}{\tau_1}) - P_2 \cdot t_3 + P_2 \cdot \tau_2 \cdot \exp(-\frac{t-t_3}{\tau_2}) - t_2 + P_1 \cdot t_2 + P_2 \cdot t_2 - P_1 \cdot \tau_1 \cdot \exp(-\frac{t-t_2}{\tau_1}) - P_2 \cdot \tau_2 \cdot \exp(-\frac{t-t_2}{\tau_2}))$$

$\sigma_4(t) := \sigma_2(t) + \sigma_{4\_1}(t)$

**Piecewise stress function**

t := t0..t4

$$\sigma(t) := \begin{cases} (\sigma_1(t)) & \text{if } 0 < t \leq t_1 \\ (\sigma_2(t)) & \text{if } t_1 < t \leq t_2 \\ (\sigma_3(t)) & \text{if } t_2 < t \leq t_3 \\ (\sigma_4(t)) & \text{if } t_3 < t \leq t_4 \\ 0 & \text{otherwise} \end{cases}$$

**Set up a matrix to print out of data**

$x_{t,0} := t \quad x_{t,1} := \epsilon(t) \quad x_{t,2} := \sigma(t)$

**Write the result file (OUTPUT.PRN)**

WRITEPRN(output) := x

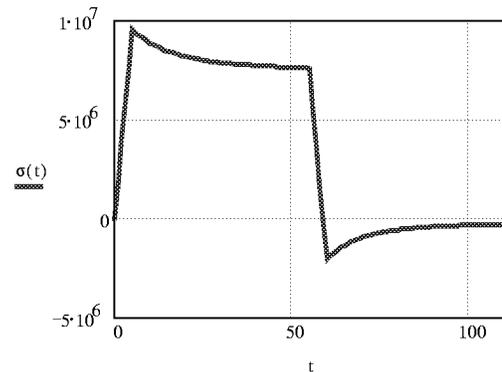
(End of Hereditary.MCD)

**Output file (OUTPUT.PRN)**

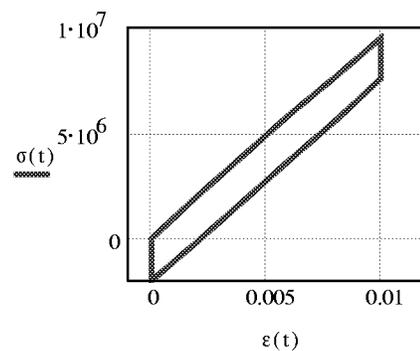
Three columns of data (time, strain and stress) are included in the output file as shown below:

(time)	(strain)	(stress)
0	0	0
1	0.002	1979653.653
2	0.004	3921103.522
3	0.006	5827816.446
4	0.008	7702931.033
5	0.01	9549288.871
6	0.01	9389809.131
7	0.01	9244678.865

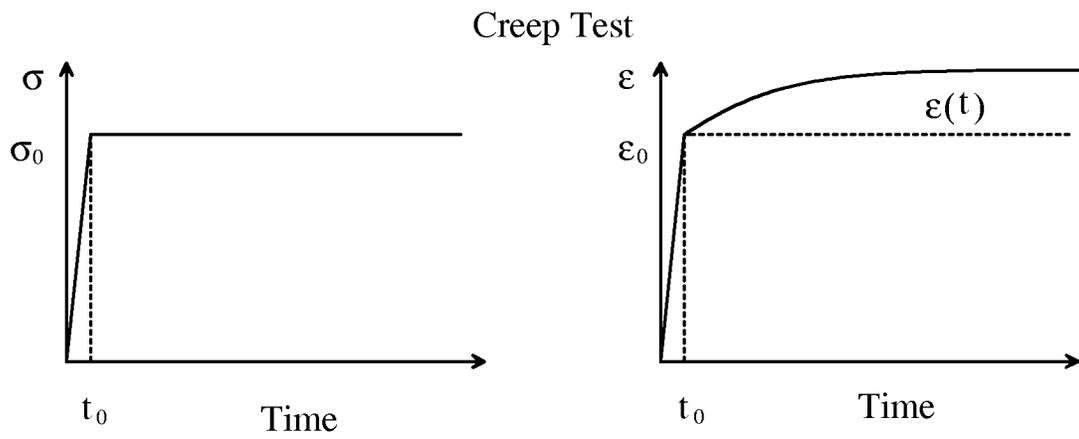
**stress function**



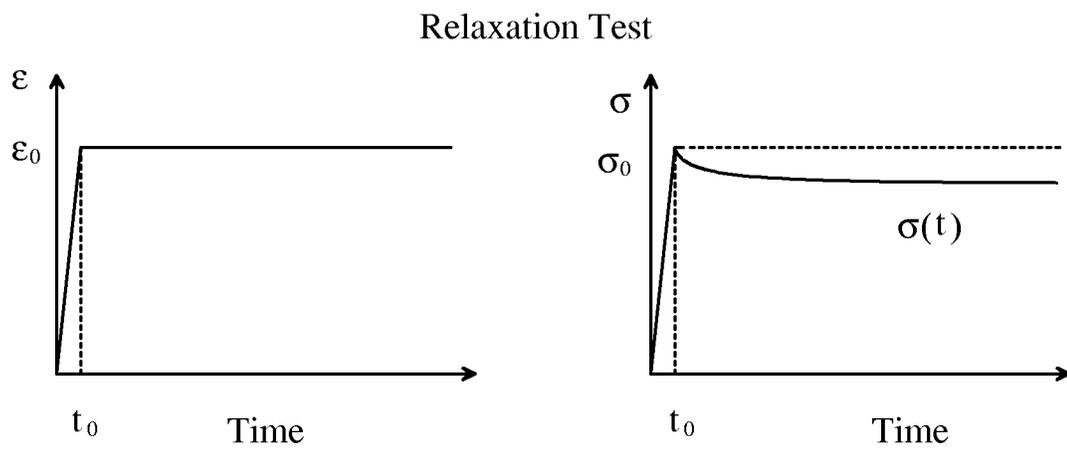
**Stress vs. Strain**



(a)



(b)



**Figure 1.** Viscoelastic material characterization tests: a) creep test, b) relaxation test.

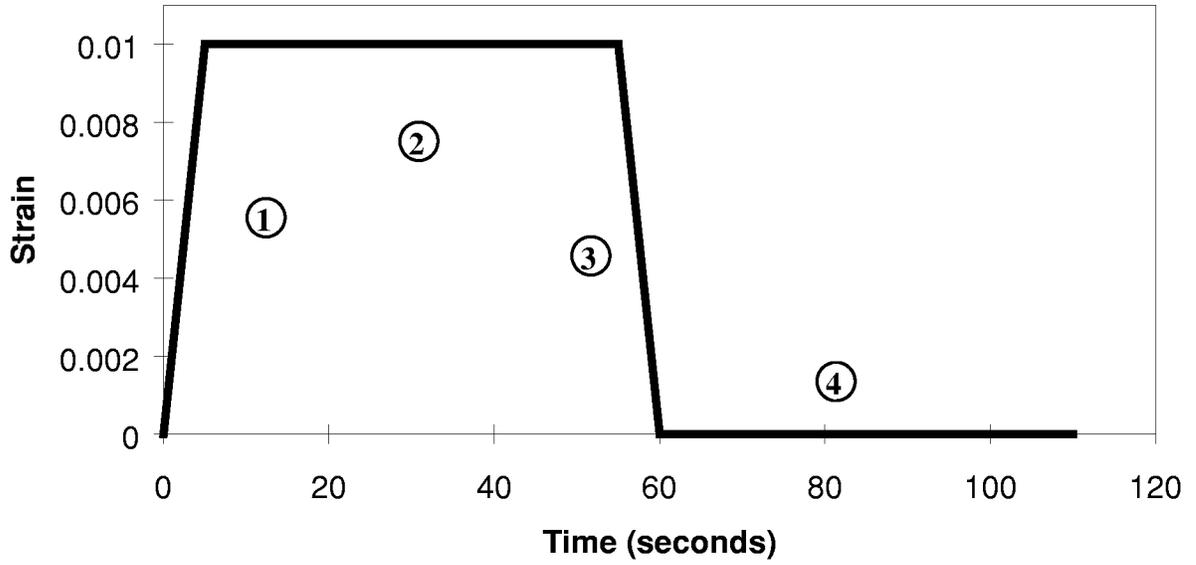


Figure 2. Multiple-segment loading process

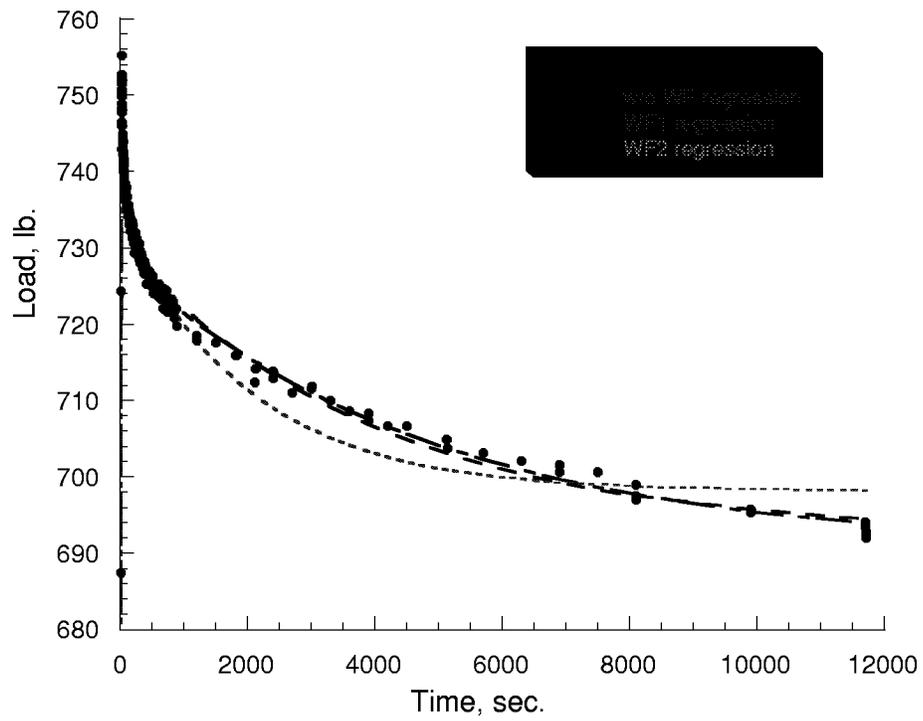
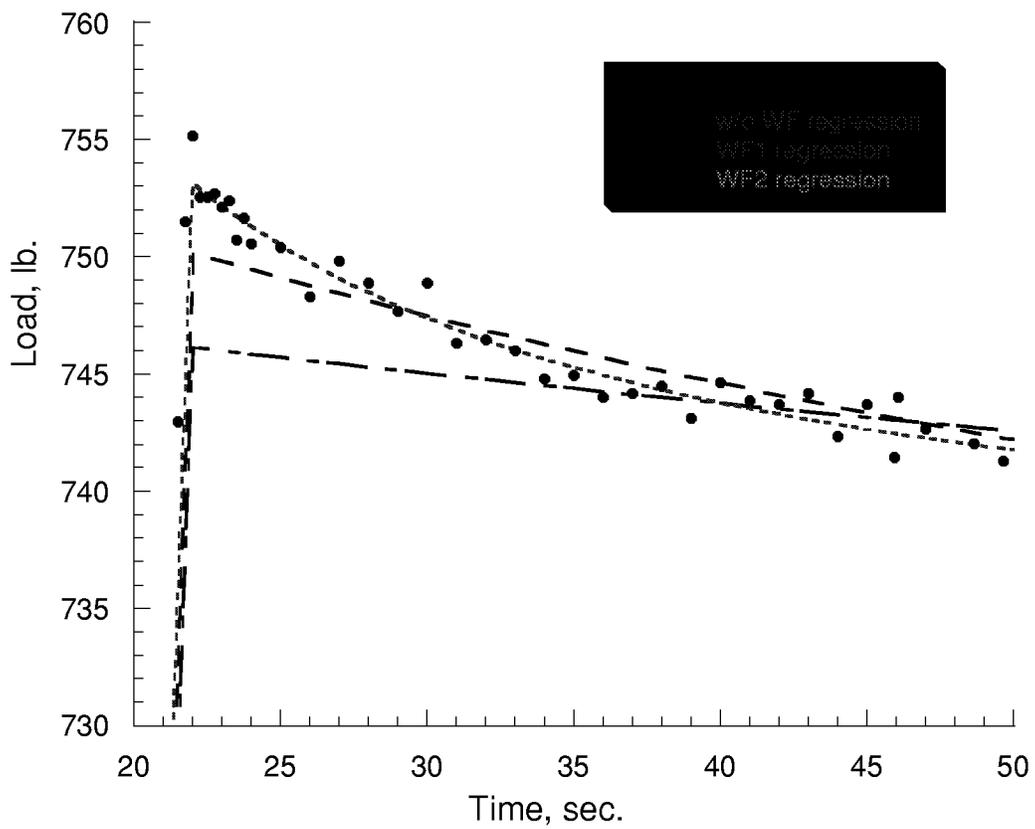
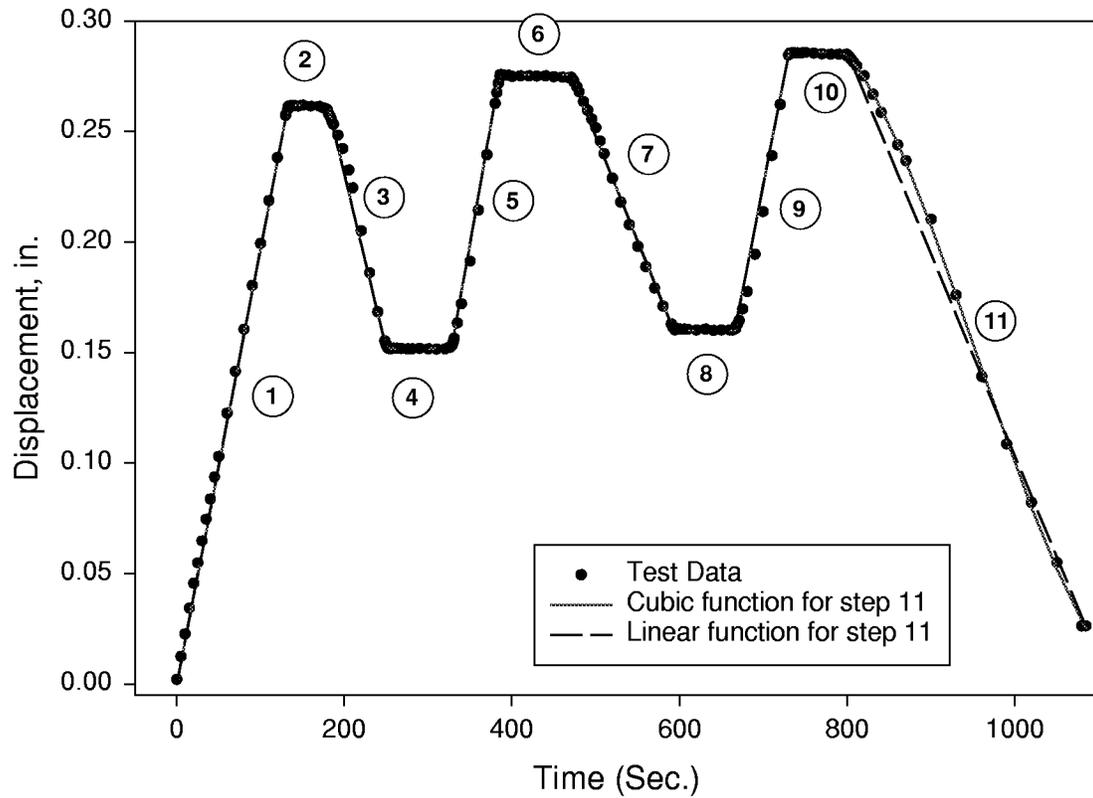


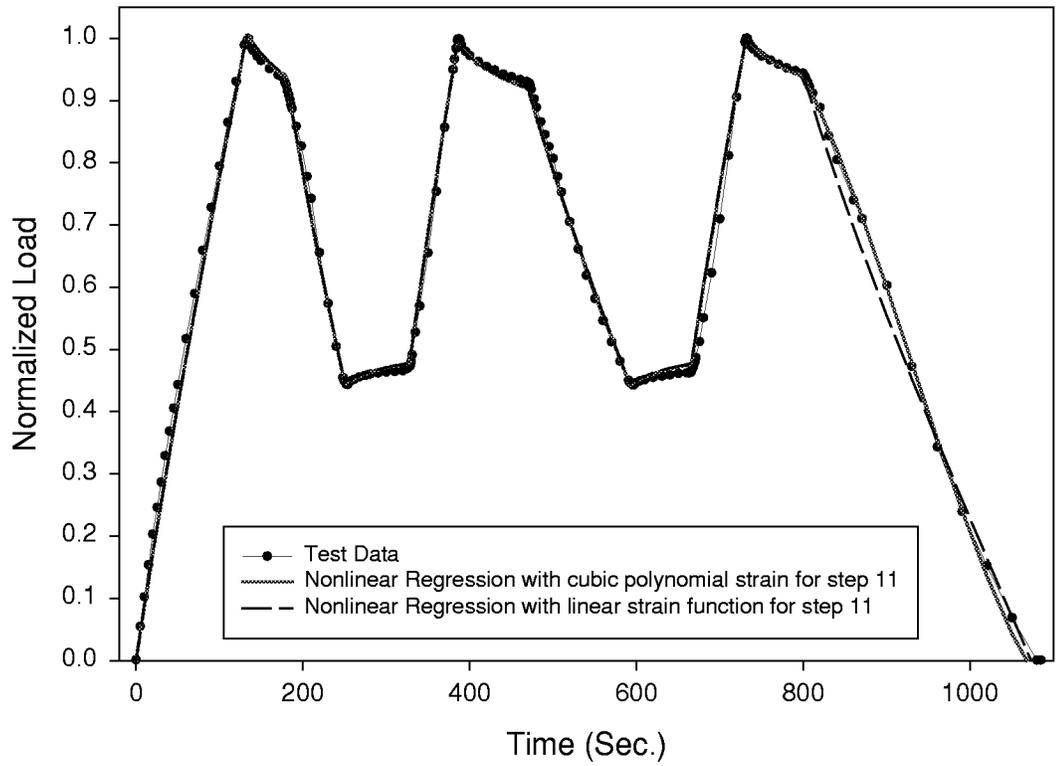
Figure 3. The regression results of relaxation test



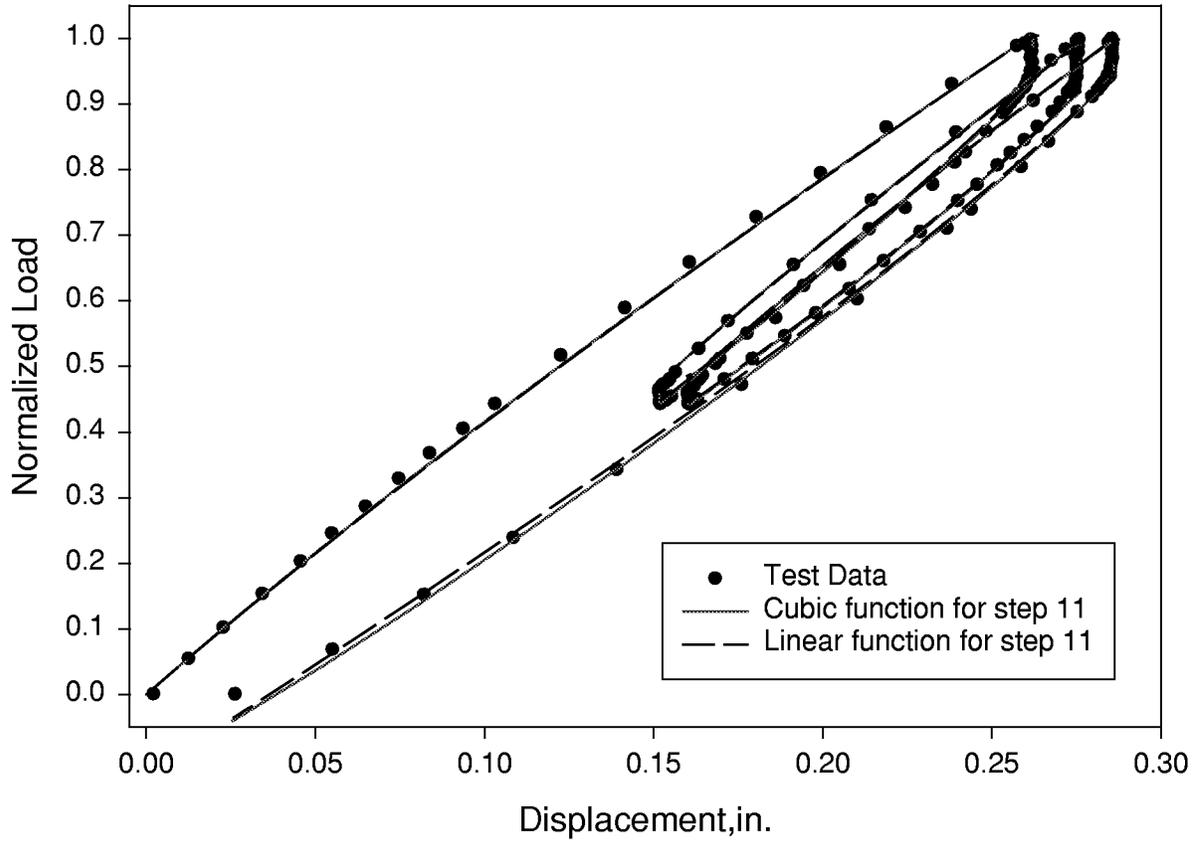
**Figure 4.** The regression results at beginning of relaxation test.



**Figure 5.** Test and simulation of loading schedule



**Figure 6.** Normalized load vs. time for the test data and nonlinear regression results.



**Figure 7.** Test and regression simulation of Normalized load-displacement for CAV panel

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE May 2000	3. REPORT TYPE AND DATES COVERED Technical Memorandum	
4. TITLE AND SUBTITLE Determining a Prony Series for a Viscoelastic Material From Time Varying Strain Data			5. FUNDING NUMBERS WU 706-13-31-01	
6. AUTHOR(S) Tzikang Chen				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, VA 23681-2199			8. PERFORMING ORGANIZATION REPORT NUMBER L-17978	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001 and U.S. Army Research Laboratory Adelphi, MD 20783-1145			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA/TM-2000-210123 ARL-TR-2206	
11. SUPPLEMENTARY NOTES				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified-Unlimited Subject Category 64                      Distribution: Nonstandard Availability: NASA CASI (301) 621-0390			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) In this study a method of determining the coefficients in a Prony series representation of a viscoelastic modulus from rate dependent data is presented. Load versus time test data for a sequence of different rate loading segments is least-squares fitted to a Prony series hereditary integral model of the material tested. A nonlinear least squares regression algorithm is employed. The measured data includes ramp loading, relaxation, and unloading stress-strain data. The resulting Prony series which captures strain rate loading and unloading effects, produces an excellent fit to the complex loading sequence.				
14. SUBJECT TERMS hereditary integral, viscoelasticity, weighted nonlinear regression, Prony series, multiple loading segments			15. NUMBER OF PAGES 26	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT UL	